# MTH 520/622 Midterm Solutions

1. Consider the groups $\text{Möb}(\mathbb{H}) = \{m \in \text{Möb}(\hat{\mathbb{C}}) : m(\mathbb{H}) = \mathbb{H}\}$ and $\text{Möb}^+(\mathbb{H}) = \{m \in \text{Möb}^+(\hat{\mathbb{C}}) : m(\mathbb{H}) = \mathbb{H}\}$.

   (a) Show that $\text{Möb}^+(\mathbb{H})$ acts transitively on $\mathbb{H}$.

   (b) Show that $\text{Möb}^+(\mathbb{H})$ acts transitively on the set of hyperbolic lines in $\mathbb{H}$.

   (c) Show that $\text{Möb}(\mathbb{H})$ acts triply transitively on the set of triples of distinct points in $\bar{\mathbb{R}}$. Does this property also hold true for the action of $\text{Möb}^+(\mathbb{H})$? Why or why not?

   **Solution.** (a) Consider a pair $z_1, z_2$ of distinct point in $\mathbb{H}$. Pick two other points $w_1, w_2 \in \mathbb{H}$ such that $d_{\mathbb{H}}(z_1, w_1) = d_{\mathbb{H}}(z_2, w_2)$. Since $\text{Möb}^+(\mathbb{H})$ acts transitively on such equidistant pairs of points preserving order, there exists $m \in \text{Möb}^+(\mathbb{H})$ such that $m(z_1) = z_2$.

   (b) It suffices to show that every hyperbolic line $L$ can be mapped to the imaginary axis $L'$ by some element $m \in \text{Möb}^+(\mathbb{H})$. Consider points $z_1, z_2 \in L$, and let $d_{\mathbb{H}}(z_1, z_2) = d$. Then there is an $m \in \text{Möb}^+(\mathbb{H})$ such that $m(z_1) = i$ and $m(z_2) = e^d i$. Since any two distinct points in $\mathbb{H}$ have a unique geodesic joining them, it follows that $m(L) = L'$.

   (c) Consider a triple $r_1, r_2, r_3$ of points distinct in $\bar{\mathbb{R}}$. It suffices to show that there exists $m \in \text{Möb}(\mathbb{H})$ that maps this triple to the triple $0, 1, \infty$. But there is a unique map $m \in \text{Möb}(\mathbb{H})$ (up to normalization) with this property, namely

$$m(z) = [z, r_3; r_2, r_1] = \frac{(z - r_1)(r_2 - r_3)}{(z - r_3)(r_2 - r_1)}.$$

   Moreover, $m \in \text{Möb}^+(\mathbb{H})$ if, and only if $\text{Det}(m) > 0$, that is, $r_1 < r_2 < r_3$. Hence, it follows that the action of $\text{Möb}^+(\mathbb{H})$ on $\bar{\mathbb{R}}$ is not triply transitive. In particular, there exists no $m \in \text{Möb}^+(\mathbb{H})$ that maps the triple $0, 1, \infty$ to the triple $0, -1, \infty$.

2. Consider an $m \in \text{Möb}^+(\mathbb{H})$ that is nontrivial. Use the classification of isometries in $\text{Möb}(\hat{\mathbb{C}})$ to prove the following.

   (a) $m$ is parabolic if, and only if $m$ has one fixed point in $\bar{\mathbb{R}}$. Furthermore, $m$ is conjugate in $\text{Möb}(\mathbb{H})$ to the map $q(z) = z + 1$.

(b) $m$ is elliptic if, and only if $m$ has one fixed point in $\mathbb{H}$. Furthermore, $m$ is conjugate in $\text{Möb}^+(\mathbb{H})$ to a rotation by $\theta$ (i.e a map of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, for some $\theta \in \mathbb{R}$).

(c) $m$ is loxodromic if, and only if $m$ has two fixed points in $\bar{\mathbb{R}}$. Furthermore, $m$ is conjugate in $\text{Möb}^+(\mathbb{H})$ to the map $q(z) = kz$, for some $k > 0$.

(d) For each of the three types of isometries, derive conditions equivalent to those given in 2(a) - (c) in terms of $\text{Trace}^2(m)$.

**Solution.** Consider an $m \in \text{Möb}^+(\mathbb{H})$ given by $m(z) = \frac{az+b}{cz+d}$. The equation $m(z) = z$ yields the quadratic equation

$$cz^2 + (d-a)z - b = 0,$$

whose discriminant $D$ is given by

$$D = (a+d)^2 - 4 = \text{Trace}^2(m) - 4. \qquad (*)$$

By the classification of Möbius transformation in $\hat{\mathbb{C}}$, we know that a Möbius transformation $m \in \text{Möb}(\hat{\mathbb{C}})$ that is not the identity is:

(i) *parabolic*, if it has only one fixed point in $\hat{\mathbb{C}}$ and is conjugate to the map $m'(z) = z + 1$. Equivalently, $m$ is parabolic if, and only if $\text{Trace}^2(m) = 4$.

(ii) *elliptic*, if it has two fixed points in $\hat{\mathbb{C}}$ and is conjugate to the map $m'(z) = az$, where $|a| = 1$, that is, $a = e^{i2\theta}$, for some $\theta \in [0, \pi)$. Equivalently, $m$ is elliptic if, and only if $\text{Trace}^2(m) \in [0, 4)$.

(iii) *loxodromic*, if it has two fixed points in $\hat{\mathbb{C}}$ and is conjugate to the map $m'(z) = az$, where $|a| \neq 1$, that is, $a = re^{i2\theta}$, for some $r > 0$ and $\theta \in [0, \pi)$. Equivalently, $m$ is loxodromic if, and only if either $\text{Im}(\text{Trace}^2(m)) \neq 0$ or $\text{Trace}^2(m) \in (-\infty, 0) \cup (4, \infty)$.

We will now apply the results in (i)-(iii) and (*) to obtain a solution to the problem.

(a) An $m \in \text{Möb}^+(\mathbb{H})$ is parabolic if, and only if $\text{Trace}^2(m) = 4$ (i.e $D = 0$). Such an $m$ will have a unique fixed point in $\bar{\mathbb{R}}$ given by $(a-d)/2$. Moreover, using the triple transitivity of $\text{Möb}(\mathbb{H})$ in $\bar{\mathbb{R}}$ (shown in 1(c) above), we can conjugate $m$ to have the form $m(z) = z + 1$ (why?), whose unique fixed point is $\infty$.

It is interesting to note that parabolic transformation $m$ also preserves any horocircle that is tangential to its unique fixed point in $\bar{\mathbb{R}}$ (why?). In particular, for the map $m(z) = z+1$, these horocircles are circles in $\hat{\mathbb{C}}$ which are unions of horizontal lines with the point at $\infty$.

(b) An $m \in \text{Möb}^+(\mathbb{H})$ is elliptic if, and only if $\text{Trace}^2(m) < 4$ (i.e $D < 0$). Such an $m$ will have a unique fixed point $w$ in $\mathbb{H}$ given by $w = \frac{1}{2}(a - d + \sqrt{D}i)$, and $m$ is a rotation by some angle $\theta$ about this point $w$. In particular, if we take the fixed point of $m$ to be at $w = i$ (why? Give an example of such a map), then by a direct calculation, we see that $a = d$ and $b = -c$, and so $m$ has the form

$$m(z) = \frac{az + b}{-bz + a}.$$

We normalize by dividing each coefficient on the right by $\sqrt{a^2 + b^2}$ to obtain the form $m(z) = e^{i\theta} z$, where $\theta = \cos^{-1}(a/\sqrt{a^2 + b^2})$, as desired in the problem.

The geometry of an elliptic map is more apparent in the Poincaré disk model, if we consider the Cayley transformation $C : \mathbb{H} \to \mathbb{D}$. As $C \in \text{Möb}^+(\hat{\mathbb{C}})$, it follows that $C \circ m \in \text{Isom}^+(\mathbb{D})$ is a rotation of $\mathbb{D}$ by $\theta$.

(c) An $m \in \text{Möb}^+(\mathbb{H})$ is loxodromic if, and only if $\text{Trace}^2(m) > 4$ (i.e $D > 0$). Such an $m$ will have two fixed points in $\bar{\mathbb{R}}$ given by $\frac{1}{2}(a - d \pm \sqrt{D})$. Furthermore, $m$ preserves the unique geodesic $L$ in $\mathbb{H}$ joining these points (also called the *axis* of the loxodromic transformation $m$). In fact, $m$ is a "glide reflection" (i.e the composition of a reflection about a line with a translation along the line) about $L$ (why? Give an example of such a map).

Suppose that $x$ and $y$ ($x < y$) are the two fixed points of $m$. Using the triple transitivity of $\text{Möb}^+(\mathbb{H})$ on ordered triples in $\bar{\mathbb{R}}$, we may assume up to conjugacy that $x = 0$ and $y = \infty$, so that the axis of $m$ is the imaginary axis (why?). Hence, up to conjugacy, $m$ has the form $m(z) = \lambda z$.

3. If $\gamma : [a, b] \to \mathbb{H}$ is a piecewise $C^1$ path, then show that $\ell_{\mathbb{H}}(\gamma) < \infty$.

**Solution.** Consider a partition of $[a,b]$ into closed subintervals $I_k$, $1 \leq k \leq r$ such that $\gamma'|_{I_k}$ is continuous. By the extreme value theorem, for $1 \leq k \leq r$, there exists $M_k$ such that

$$|\gamma'(x)| \leq M_k, \text{ for } x \in I_k.$$

Let $M = \max_k M_k$. Now using the fact that there exists $N > 0$ such that $\operatorname{Im}(\gamma(t)) \geq N$, for all $t \in [a,b]$ (why?), we have

$$\ell_{\mathbb{H}}(\gamma) = \int_a^b \frac{1}{\operatorname{Im}(\gamma(t))} |\gamma'(t)| dt \leq \frac{M}{N}(b-a).$$

4. **(Bonus)** Show that the subspace topology induced on $\mathbb{H}$ by the metric topology on $\mathbb{R}^2$ is identical to the metric topology on $(\mathbb{H}, d_{\mathbb{H}})$.

   **Solution.** This follows from assertion 3.3(v) of the lesson plan, which was discussed in class. Fill in the details.